Int. J. Solids Structures, 1968, Vol. 4, pp. 845 to 855. Pergamon Press. Printed in Great Britain

ON THE THEORY OF PLANE STRESS FOR FINITE DEFORMATIONS*

SEH-IEH CHOU

Department of Engineering Mechanics, University of Nebraska, Lincoln, Nebraska

Abstract—By means of parametric expansions in terms of thickness h_0 of the undeformed plate, the equations of plane stress for finite deformations of homogeneous, isotropic, elastic materials are derived from the three-dimensional theory of finite elasticity without making any usual geometrical or physical assumptions other than that deformation is symmetrical with respect to the middle plane of the plate. The equations of plane stress are obtained as the coefficients of zero power of h_0 in the expansions. Coefficients of higher order terms provide interior corrections to the plane stress theory. The appropriate boundary conditions for these interior equations are derived by the variational formulation of the three-dimensional theory of finite elasticity.

1. INTRODUCTION

THE general theory of plane stress for finite deformations of isotropic, elastic materials has been considered by Adkins *et al.* [1] and can be found in [2]. The equations are derived by assuming the thickness of the plate to be small so that, when no forces act on the major surfaces of the plate, it is assumed that the principal stress components normal to the middle plane of the plate vanishes everywhere, as in the classical theory of plane stress. Instead of examining stresses at every point across the thickness of the plate, the stress resultants as functions of position on the middle surface of the plate are considered.

In this paper, the equations of plane stress and boundary conditions for finite deformations are derived systematically from the three-dimensional theory of elasticity. The method consists in assuming that the deformation is symmetrical with respect to the middle plane of the plate and expanding the deformations and stresses into powers of the thickness h_0 of the undeformed plate. Substituting these expansions into the constitutive relations and equations of equilibrium, successive systems of equations are obtained by equating terms of like powers of h_0. The lowest order terms correspond to the equations of plane stress theory, whereas higher order terms provide interior corrections to the plane stress theory. The appropriate boundary conditions for these interior equations are derived by the variational formulation of the three-dimensional theory of finite elasticity, without consideration of the edge-zone solution. A similar problem for the classical linear theory of plane stress has been considered by Reiss and Locke [3]. They derived the equations of plane stress and boundary conditions by simultaneously considering expansions of interior and edge-zone solutions, which is a generalization of the boundary layer method used by Friedrichs and Dressler [4].

* The results presented in this paper were obtained in the course of research supported by the College of Engineering and Architecture of the University of Nebraska through a Faculty Summer Research Fellowship, 1966.

2. FORMULATION OF THE PROBLEM

Let a particle in the undeformed state be denoted by x_i and the same particle after deformation be denoted by y_i, which is referred to the same set of rectangular Cartesian coordinates as x_i. We consider a plate of constant thickness h_0 in its undeformed state bounded by the planes $x_3 = \pm h_0$ and the cylindrical surface $f(x_1, x_2) = 0$. The bounding planes $x_3 = \pm h_0$ are referred to as the surface boundaries of the undeformed plate. The surface $|x_3| \leq h_0$, $f(x_1, x_2) = 0$ is called the edge boundary of the undeformed plate. The smooth curve B; $x_1 = x_1(s)$, $x_2 = x_2(s)$, $x_3 = 0$, is called the boundary curve of the undeformed plate.

We assume that the deformation is described by the mapping

$$y_i = y_i(x_1, x_2, x_3). \tag{2.1}$$

Using the formulas given by Green and Adkins [2], the strain tensor E_{ij} is defined by

$$E_{ij} = \tfrac{1}{2}(C_{ij} - \delta_{ij}) \tag{2.2}$$

where

$$C_{ij} = y_{r,i} y_{r,j} \tag{2.3}$$

and δ_{ij} is the Kronecker delta. The usual summation convention is used and Latin indices take the values 1, 2 and 3. A comma denotes partial differentiation with respect to x_i.

The equations of equilibrium in the absence of body forces can be written as

$$(s_{jk} y_{i,k})_{,j} = 0 \tag{2.4}$$

where s_{ij} is the symmetric stress tensor in the deformed body measured per unit area of the undeformed body.

When the body is homogeneous and isotropic, the strain energy function W per unit area of the undeformed body is

$$W = W(I_1, I_2, I_3)$$

where I_1, I_2 and I_3 are strain invariants

$$I_1 = C_{ii}, \qquad I_2 = (I_1^2 - C_{mn}C_{mn})/2, \qquad I_3 = \det(C_{ij}). \tag{2.5}$$

The constitutive relations take the form

$$s_{ij} = \frac{\partial W}{\partial E_{ij}} \tag{2.6}$$

or may also be written as

$$s_{ij} = \Phi \delta_{ij} + \Psi B_{ij} + p\bar{C}_{ij} \tag{2.7}$$

where

$$\Phi = 2\frac{\partial W}{\partial I_1}, \qquad \Psi = 2\frac{\partial W}{\partial I_2}, \qquad p = 2I_3\frac{\partial W}{\partial I_3} \tag{2.8}$$

and

$$B_{ij} = \delta_{ij} I_1 - C_{ij} \tag{2.9}$$

In (2.7), $\bar{C}_{ij}$ is the inverse of C_{ij} and has the relation

$$C_{ik}\bar{C}_{kj} = \delta_{ij}. \tag{2.10}$$

When the material is incompressible, $I_3 = 1$ and W is a function of I_1 and I_2 only. The constitutive relations (2.7) still hold, but in this case p is a scalar invariant function of x_i.

When surface forces ${}_0t_j^*$ are prescribed at the boundary

$$s_{ki}y_{j,k}\,{}_0n_i = {}_0t_j^* \tag{2.11}$$

where ${}_0n_j$ is the component of unit normal vector to the undeformed position of a surface in the deformed body. On the surface boundaries $x_3 = \pm h_0$, the unit normal vector is given by

$${}_0n_i = (0, 0, 1).$$

Assuming that surface boundaries are free from surface tractions, (2.11) then yields

$$s_{k3}y_{j,k} = 0$$

which has a unique solution

$$s_{k3} = 0 \quad \text{at } x_3 = \pm h_0 \tag{2.12}$$

since $\det(y_{j,k})$ is assumed to be nonvanishing. On the edge surface, the unit normal vector is given by

$${}_0n_j = ({}_0n_1, {}_0n_2, 0).$$

The edge boundary conditions are, from (2.11),

$$s_{k\alpha}y_{j,k}\,{}_0n_\alpha = {}_0t_j^* \quad \text{on } B. \tag{2.13}$$

Here ${}_0n_\alpha$ is the component of unit normal vector on B. Greek indices here and in the following take the values 1 and 2.

Although any deformation can be considered as a sum of symmetrical and antisymmetrical deformations satisfying coupled equations, in this paper attention will be given to deformation symmetrical with respect to the middle plane, i.e., extension of the plate by edge forces. This implies that y_α, $s_{\alpha\beta}$, s_{33} are even functions of x_3, while y_3, $s_{\alpha 3}$, are odd functions of x_3.

3. PARAMETRIC EXPANSIONS

It is assumed that y_α, which is an even function of x_3, and y_3, which is an odd function of x_3, may be expanded into powers of x_3 in the form

$$y_\alpha(x_1, x_2, x_3) = y_\alpha^{(0)}(x_1, x_2, 0) + x_3^2 y_\alpha^{(2)}(x_1, x_2, 0) + x_3^4 y_\alpha^{(4)}(x_1, x_2, 0) + \cdots \tag{3.1}$$

$$y_3(x_1, x_2, x_3) = x_3 y_3^{(0)}(x_1, x_2, 0) + x_3^3 y_3^{(2)}(x_1, x_2, 0) + \cdots \tag{3.2}$$

Introducing nondimensional quantities

$$\xi = x_3/h_0, \qquad Z = y_3/h_0 \tag{3.3}$$

expressions (3.1) and (3.2) then yield

$$y_\alpha(x_1, x_2, \xi; h_0) = y_\alpha^{(0)}(x_1, x_2) + h_0^2\xi^2 y_\alpha^{(2)}(x_1, x_2) + h_0^4\xi^4 y_\alpha^{(4)}(x_1, x_2) + \cdots \tag{3.4}$$

$$Z(x_1, x_2, \xi; h_0) = \xi Z^{(0)}(x_1, x_2) + h_0^2\xi^3 Z^{(2)}(x_1, x_2) + \cdots \tag{3.5}$$

where

$$y_\alpha^{(n)}(x_1, x_2) = y_\alpha^{(n)}(x_1, x_2, 0)$$
$$Z^{(n)}(x_1, x_2) = y_3^{(n)}(x_1, x_2, 0)$$

are functions of x_α only.

We also assume that the tensors C_{ij}, $\bar{C}_{ij}$, s_{ij}, B_{ij} and strain invariants I_1, I_2, I_3 can be expanded into powers of h_0 in the form

$$f(x_1, x_2, \xi; h_0) = f^{(0)}(x_1, x_2, \xi) + h_0 f^{(1)}(x_1, x_2, \xi) + h_0^2 f^{(2)}(x_1, x_2, \xi) + \cdots. \tag{3.6}$$

Substituting expansion of the form given by (3.6) into (2.3) and equating terms of like powers, we have

$$\begin{aligned}
C_{\alpha\beta}^{(0)} &= y_{\rho,\alpha}^{(0)} y_{\rho,\beta}^{(0)}, \qquad C_{\alpha\beta}^{(1)} = C_{\alpha\beta}^{(3)} = C_{\alpha\beta}^{(5)} = \cdots = 0 \\
C_{\alpha\beta}^{(2)} &= 6^2(y_{\rho,\alpha}^{(0)} y_{\rho,\beta}^{(2)} + y_{\rho,\beta}^{(2)} y_{\rho,\alpha}^{(0)} + Z_{,\alpha}^{(0)} Z_{,\beta}^{(0)}) \\
C_{\alpha\beta}^{(4)} &= \cdots, \\
C_{\alpha 3}^{(0)} &= C_{\alpha 3}^{(2)} = C_{\alpha 3}^{(4)} = \cdots = 0 \\
C_{\alpha 3}^{(1)} &= \xi(Z_{,\alpha}^{(0)} Z^{(0)} + 2y_{\rho,\alpha}^{(0)} y_\rho^{(2)}) \\
C_{\alpha 3}^{(3)} &= \xi^3(2y_{\rho,\alpha}^{(2)} y_\rho^{(2)} + 4y_{\rho,\alpha}^{(0)} y_\rho^{(4)} + Z_{,\alpha}^{(2)} Z^{(0)} + 3Z_{,\alpha}^{(0)} Z^{(2)}) \\
C_{\alpha 3}^{(5)} &= \cdots, \\
C_{33}^{(0)} &= (Z^{(0)})^2, \qquad C_{33}^{(1)} = C_{33}^{(3)} = \cdots = 0 \\
C_{33}^{(2)} &= 2\xi^2(2y_\rho^{(2)} y_\rho^{(2)} + 3Z^{(0)} Z^{(2)}) \\
C_{33}^{(4)} &= \cdots.
\end{aligned} \tag{3.7}$$

Equation (2.10) can be written as

$$(C_{ir}^{(0)} + h_0 C_{ir}^{(1)} + h_0^2 C_{ir}^{(2)} + \cdots)(\bar{C}_{rj}^{(0)} + h_0 \bar{C}_{rj}^{(1)} + h_0^2 \bar{C}_{rj}^{(2)} + \cdots) = \delta_{ij}$$

which yields

$$\begin{aligned}
C_{ir}^{(0)} \bar{C}_{rj}^{(0)} &= \delta_{ij} \\
C_{ir}^{(1)} \bar{C}_{rj}^{(0)} + C_{ir}^{(0)} \bar{C}_{rj}^{(1)} &= 0 \\
C_{ir}^{(0)} \bar{C}_{rj}^{(2)} + C_{ir}^{(1)} \bar{C}_{rj}^{(1)} + C_{ir}^{(2)} \bar{C}_{rj}^{(0)} &= 0 \\
C_{ir}^{(0)} \bar{C}_{rj}^{(3)} + C_{ir}^{(1)} \bar{C}_{rj}^{(2)} + C_{ir}^{(2)} \bar{C}_{rj}^{(1)} + C_{ir}^{(3)} \bar{C}_{rj}^{(0)} &= 0, \text{ etc.}
\end{aligned} \tag{3.8}$$

The solutions of (3.8) for $\bar{C}_{ij}^{(n)}$ yield

$$
\begin{aligned}
C_{\alpha\gamma}^{(0)}\bar{C}_{\gamma\beta}^{(0)} &= \delta_{\alpha\beta}, \qquad \bar{C}_{33}^{(0)} = 1/C_{33}^{(0)}, \qquad \bar{C}_{\alpha 3}^{(0)} = 0 \\
\bar{C}_{\alpha\beta}^{(1)} &= \bar{C}_{33}^{(1)} = 0, \qquad \bar{C}_{\alpha 3}^{(1)} = -C_{\eta 3}^{(1)}\bar{C}_{33}^{(0)}\bar{C}_{\alpha\eta}^{(0)} \\
\bar{C}_{\alpha\beta}^{(2)} &= -C_{\eta 3}^{(1)}\bar{C}_{3\beta}^{(1)}\bar{C}_{\alpha\eta}^{(0)} - C_{\eta\gamma}^{(2)}\bar{C}_{\gamma\beta}^{(0)}\bar{C}_{\alpha\eta}^{(0)} \\
\bar{C}_{33}^{(2)} &= -(C_{3\gamma}^{(1)}\bar{C}_{\gamma 3}^{(1)} + C_{33}^{(2)}\bar{C}_{33}^{(0)})/C_{33}^{(0)}, \qquad \bar{C}_{\alpha 3}^{(2)} = 0 \\
\bar{C}_{\alpha\beta}^{(3)} &= \bar{C}_{33}^{(3)} = 0 \\
\bar{C}_{\alpha 3}^{(3)} &= -C_{\eta 3}^{(1)}\bar{C}_{33}^{(2)}\bar{C}_{\alpha\eta}^{(0)} - C_{\eta\gamma}^{(2)}\bar{C}_{\gamma 3}^{(1)}\bar{C}_{\alpha\eta}^{(0)} - C_{\eta 3}^{(3)}\bar{C}_{33}^{(0)}\bar{C}_{\alpha\eta}^{(0)} \\
\bar{C}_{\alpha\beta}^{(4)} &= \cdots.
\end{aligned} \tag{3.9}
$$

Note that each term in (3.9) can be determined step by step starting with the first equation of (3.9) with the help of (3.7), since the right-hand side of (3.9) are known functions at each step. The strain invariants, from (2.5), yield

$$
\begin{aligned}
I_1^{(0)} &= C_{ii}^{(0)}, \qquad I_1^{(1)} = I_1^{(3)} = \cdots = 0 \\
I_1^{(2)} &= C_{ii}^{(2)}, \qquad I_1^{(4)} = C_{ii}^{(4)}, \cdots \\
I_2^{(0)} &= [(I_1^{(0)})^2 - C_{\alpha\beta}^{(0)}C_{\alpha\beta}^{(0)} - (C_{33}^{(0)})^2]/2, \\
I_2^{(1)} &= I_1^{(3)} = \cdots = 0 \\
I_1^{(2)} &= I_1^{(0)}I_1^{(2)} - C_{\alpha\beta}^{(0)}C_{\alpha\beta}^{(2)} - C_{33}^{(0)}C_{33}^{(2)} - C_{\alpha 3}^{(1)}C_{\alpha 3}^{(1)}/2 \\
I_2^{(4)} &= \cdots, \\
I_3^{(0)} &= [C_{11}^{(0)}C_{22}^{(0)} - (C_{12}^{(0)})^2]C_{33}^{(0)}, \qquad I_3^{(1)} = I_3^{(3)} = \cdots = 0 \\
I_3^{(2)} &= C_{11}^{(0)}C_{22}^{(0)}C_{33}^{(2)} + C_{22}^{(0)}C_{33}^{(0)}C_{11}^{(2)} + C_{33}^{(0)}C_{11}^{(0)}C_{22}^{(2)} \\
&\quad - (C_{13}^{(1)})^2C_{22}^{(0)} - (C_{23}^{(1)})^2C_{11}^{(0)} + 2C_{12}^{(0)}C_{23}^{(1)}C_{13}^{(1)} - 2C_{12}^{(0)}C_{12}^{(2)}C_{33}^{(0)}.
\end{aligned} \tag{3.10}
$$

The tensor B_{ij} defined by (2.9) is similarly expanded into the form given by (3.6) to yield

$$
\begin{aligned}
B_{\alpha\beta}^{(0)} &= \delta_{\alpha\beta}I_1^{(0)} - C_{\alpha\beta}^{(0)}, \qquad B_{\alpha\beta}^{(1)} = B_{\alpha\beta}^{(3)} = \cdots = 0 \\
B_{\alpha\beta}^{(2)} &= \delta_{\alpha\beta}I_1^{(2)} - C_{\alpha\beta}^{(2)}, \qquad B_{\alpha\beta}^{(4)} = \cdots, \\
B_{\alpha 3}^{(0)} &= B_{\alpha 3}^{(2)} = \cdots = 0 \\
B_{\alpha 3}^{(1)} &= -C_{\alpha 3}^{(1)}, \qquad B_{\alpha 3}^{(3)} = -C_{\alpha 3}^{(3)}, \cdots \\
B_{33}^{(0)} &= I_1^{(0)} - C_{33}^{(0)}, \qquad B_{33}^{(1)} = B_{33}^{(3)} = \cdots = 0 \\
B_{33}^{(2)} &= I_1^{(2)} - C_{33}^{(2)}, \qquad B_{33}^{(4)} = \cdots.
\end{aligned} \tag{3.11}
$$

The strain energy function W can be written as

$$
W = W(I_1^{(0)} + h_0^2 I_1^{(2)} + \cdots, \qquad I_2^{(0)} + h_0^2 I_2^{(2)} + \cdots, \qquad I_3^{(0)} + h_0^2 I_3^{(2)} + \cdots). \tag{3.12}
$$

The three invariant functions Φ, Ψ, and p given by (2.8) are then expanded into Taylor's series with respect to $I_1^{(0)}$, $I_2^{(0)}$ and $I_3^{(0)}$ to yield,

$$
\begin{aligned}
&\Phi_0 = \Phi_0(I_1^{(0)}, I_2^{(0)}, I_3^{(0)}), \qquad \Phi_1 = \Phi_3 = \Phi_5 = \cdots = 0 \\
&\Phi_2 = BI_1^{(2)} + CI_2^{(2)} + DI_3^{(2)} \\
&\Phi_4 = \cdots, \\
&\Psi_0 = \Psi_0(I_1^{(0)}, I_2^{(0)}, I_3^{(0)}), \qquad \Psi_1 = \Psi_3 = \Psi_5 = \cdots = 0 \\
&\Psi_2 = CI_1^{(2)} + EI_2^{(2)} + FI_3^{(2)} \\
&\Psi_4 = \cdots, \\
&p_0 = p_0(I_1^{(0)}, I_2^{(0)}, I_3^{(0)}), \qquad p_1 = p_3 = p_5 = \cdots = 0 \\
&p_2 = I_3^{(0)}(DI_1^{(2)} + FI_2^{(2)} + HI_3^{(2)}) + I_3^{(2)} p_0 / I_3^{(0)} \\
&p_4 = \cdots,
\end{aligned}
\tag{3.13}
$$

where

$$
B = 2\frac{\partial^2 W}{\partial I_1^2}, \qquad C = 2\frac{\partial^2 W}{\partial I_1 \partial I_2}, \qquad D = 2\frac{\partial^2 W}{\partial I_1 \partial I_3}
$$

$$
E = 2\frac{\partial^2 W}{\partial I_2^2}, \qquad F = 2\frac{\partial^2 W}{\partial I_2 \partial I_3}, \qquad H = 2\frac{\partial^2 W}{\partial I_3^2}
$$

are to be evaluated at $I_1 = I_1^{(0)}$, $I_2 = I_2^{(0)}$, $I_3 = I_3^{(0)}$.

The stress tensor s_{ij} is also expanded into powers of h_0 and with (3.9), (3.11), (3.13), the expansion yields

$$
\left.
\begin{aligned}
s_{\alpha\beta}^{(0)} &= \Phi_0 \delta_{\alpha\beta} + \Psi_0[\delta_{\alpha\beta}(C_{\rho\rho}^{(0)} + Z^{(0)}Z^{(0)}) - C_{\alpha\beta}^{(0)}] + p_0 \bar{C}_{\alpha\beta}^{(0)} \\
s_{33}^{(0)} &= \Phi_0 + \Psi_0 C_{\rho\rho}^{(0)} + p_0 \bar{C}_{33}^{(0)} \\
s_{\alpha 3}^{(0)} &= 0
\end{aligned}
\right\}
\tag{3.14}
$$

$$
\left.
\begin{aligned}
s_{\alpha\beta}^{(1)} &= s_{33}^{(1)} = 0 \\
s_{\alpha 3}^{(1)} &= -\Psi_0 C_{\alpha 3}^{(1)} + p_0 \bar{C}_{\alpha 3}^{(1)}
\end{aligned}
\right\}
\tag{3.15}
$$

$$
\left.
\begin{aligned}
s_{\alpha\beta}^{(2)} &= \Phi_2 \delta_{\alpha\beta} + \Psi_2(\delta_{\alpha\beta} I_1^{(0)} - C_{\alpha\beta}^{(0)}) + \Psi_0(\delta_{\alpha\beta} I_1^{(2)} - C_{\alpha\beta}^{(2)}) + p_2 \bar{C}_{\alpha\beta}^{(0)} \\
s_{33}^{(2)} &= \Phi_2 + \Psi_2(I_1^{(0)} - C_{\alpha\beta}^{(0)}) + \Psi_0(I_1^{(2)} - C_{33}^{(2)}) + p_2 \bar{C}_{33}^{(0)} + p_0 \bar{C}_{33}^{(2)} \\
s_{\alpha 3}^{(2)} &= 0
\end{aligned}
\right\}
\tag{3.16}
$$

and higher order terms.

4. INTERIOR EQUATIONS

The surface boundary conditions at $\xi = \pm 1$ given by (2.12) yield

$$
s_{i3}^{(n)} = 0 \quad \text{at } \xi = \pm 1, \qquad n = 0, 1, 2, \ldots. \tag{4.1}
$$

Examining expansions (3.14)–(3.16) and with the help of (3.7), (3.9), (3.10) and (3.13), it can be shown that

$$\begin{aligned} s_{\alpha 3}^{(n)} &= \xi^n \bar{s}_{\alpha 3}^{(n)}(x_1, x_2), \qquad n = 1, 3, 5, \ldots \\ s_{33}^{(n)} &= \xi^n \bar{s}_{33}^{(n)}(x_1, x_2), \qquad n = 0, 2, 4, \ldots \end{aligned} \tag{4.2}$$

where $\bar{s}_{\alpha 3}^{(n)}, \bar{s}_{33}^{(n)}$ are functions of x_1 and x_2 only. The immediate consequence of (4.1) and (4.2) is that

$$s_{\alpha 3}^{(n)} = s_{33}^{(n)} \equiv 0, \qquad n = 0, 1, 2, \ldots \tag{4.3}$$

in the interior region of the plate.

The equations of equilibrium (2.4), with (3.1), (3.2) and (3.14)–(3.16), and (4.3) then yield, for the coefficient of zero power of h_0,

$$(s_{\alpha\beta}^{(0)} y_{\gamma,\beta}^{(0)})_{,\alpha} = 0 \tag{4.4}$$

$$(s_{\alpha\beta}^{(0)} Z_{,\beta}^{(0)})_{,\alpha} = 0. \tag{4.5}$$

Since $\Phi_0, \Psi_0, p_0, C_{\alpha\beta}^{(0)}, \bar{C}_{\alpha\beta}^{(0)}$ are all functions of x_1 and x_2 only, the first equation of (3.14) shows that $s_{\alpha\beta}^{(0)}$ is independent of ξ, i.e.

$$\begin{aligned} s_{\alpha\beta}^{(0)} &= \bar{s}_{\alpha\beta}^{(0)}(x_1, x_2) \\ &= \Phi_0 \delta_{\alpha\beta} + \Psi_0[\delta_{\alpha\beta}(C_{\rho\rho}^{(0)} + Z^{(0)}Z^{(0)}) - C_{\alpha\beta}^{(0)}] + p_0 \bar{C}_{\alpha\beta}^{(0)}. \end{aligned} \tag{4.6}$$

Equations (4.4)–(4.6) are thus six equations for the determination of six unknowns $s_{\alpha\beta}^{(0)}$, $y_\alpha^{(0)}$, $Z^{(0)}$. These are basic equations of plane stress theory.

For the coefficient of h_0^2, the equations of equilibrium similarly yield

$$(\xi^2 s_{\alpha\beta}^{(0)} y_{\gamma,\beta}^{(2)} + s_{\alpha\beta}^{(2)} y_{\gamma,\beta}^{(0)})_{,\alpha} = 0 \tag{4.7}$$

$$(\xi^2 s_{\alpha\beta}^{(0)} Z_{,\beta}^{(2)} + s_{\alpha\beta}^{(2)} Z_{,\beta}^{(0)})_{,\alpha} = 0 \tag{4.8}$$

The first equation of (3.16) shows that

$$\begin{aligned} s_{\alpha\beta}^{(2)} &= \xi^2 \bar{s}_{\alpha\beta}^{(2)}(x_1, x_2) \\ &= \Phi_2 \delta_{\alpha\beta} + \Psi_2(\delta_{\alpha\beta} I_1^{(0)} - C_{\alpha\beta}^{(0)}) + \Psi_0(\delta_{\alpha\beta} I_1^{(2)} - C_{\alpha\beta}^{(2)}) + p_2 \bar{C}_{\alpha\beta}^{(0)}. \end{aligned} \tag{4.9}$$

Equations (4.7)–(4.9) are six equations for the determination of six unknowns $s_{\alpha\beta}^{(2)}$, $y_\alpha^{(2)}$, $Z^{(2)}$, once the plane stress problem is solved. These are the first order interior corrections to the plane stress theory.

Higher order corrections can be similarly obtained by taking coefficients of higher order terms in the expansions.

5. BOUNDARY CONDITIONS

In this section, the appropriate boundary conditions for the interior equations obtained in the previous section will be derived by variational formulation of the three-dimensional theory of finite elasticity without considering edge-zone solutions. A similar method was used by Reissner [5] to derive the boundary conditions for the classical plate theory.

Consider the functional [6]

$$J = \int_{v_0} [W(E_{ij}) - s_{ij}E_{ij}]\, dv + \frac{1}{2}\int_{v_0} s_{ij}(C_{ij} - \delta_{ij})\, dv - \int_{A_e} y_i\, {}_0t_i^*\, dx_3\, ds \tag{5.1}$$

where v_0 is the undeformed plate, A_e is the edge surface of the plate, ds the element of arc length of the boundary curve B. It can be shown that [6] the vanishing of the first variation of (5.1) yields as its Euler equations strain tensor (2.2), constitutive relations (2.6), the equations of equilibrium (2.4), and the edge boundary conditions (2.13).

The first variation $\delta J = 0$ has the form

$$\begin{aligned} &\int_{v_0} \left(\frac{\partial W}{\partial E_{ij}} - s_{ij}\right) \delta E_{ij}\, dx_1\, dx_2\, dx_3 + \int_{v_0} [\tfrac{1}{2}(C_{ij} - \delta_{ij}) - E_{ij}]\, \delta s_{ij}\, dx_1\, dx_2\, dx_3 \\ &\quad - \int_{v_0} (s_{ij}y_{k,i})_{,j}\, \delta y_k\, dx_1\, dx_2\, dx_3 + \int_{A_f} [s_{i3}y_{k,i}\, \delta y_k]_{-h_0}^{h_0}\, dx_1\, dx_2 \\ &\quad + \int_{A_e} (s_{k\alpha}y_{i,k}\, {}_0n_\alpha - {}_0t_i^*)\, \delta y_i\, dx_3\, ds = 0 \end{aligned} \tag{5.2}$$

where A_f denotes the surface boundary of the undeformed plate. Equation (5.2) is to be used in conjunction with the parametric expansions given in Section 3 which satisfy the strain tensor (2.2), constitutive relation (2.7), the equations of equilibrium and the surface boundary conditions at $x = \pm h_0$. This implies that introduction of the results in Section 3 leaves the variational equation the form

$$\int_{A_e} (s_{k\alpha}y_{i,k}\, {}_0n_\alpha - {}_0t_i^*)\, \delta y_i\, dx_3\, ds = 0. \tag{5.3}$$

Introducing (3.3) and remembering that $s_{3\alpha} = 0$, equation (5.3) can be written as

$$\int_{A_e} (s_{\gamma\alpha}y_{\beta,\gamma}\, {}_0n_\alpha - {}_0t_\beta^*)\, \delta y_\beta\, d\xi\, ds + h_0 \int_{A_e} (h_0 s_{\gamma\alpha} Z_{,\gamma}\, {}_0n_\alpha - {}_0t_3^*)\, \delta Z\, d\xi\, ds = 0 \tag{5.4}$$

Using the parametric expansions of s_{ij}, y_α and Z, and assuming that ${}_0t_\beta^*$ and ${}_0t_3^*$ can be expanded into powers of h_0 in the form

$$\begin{aligned} {}_0t_\beta^* &= t_\beta^{(0)}(s, \xi) + h_0^2 t_\beta^{(2)}(s, \xi) + \cdots \\ {}_0t_3^* &= h_0 t_3^{(1)}(s, \xi) + h_0^3 t_3^{(2)}(s, \xi) + \cdots \end{aligned} \tag{5.5}$$

equation (5.4) then yields, for the coefficient of zero powers of h_0,

$$\int_{A_e} (s_{\alpha\gamma}^{(0)} y_{\beta,\gamma}^{(0)}\, {}_0n_\alpha - t_\beta^{(0)})\, \delta y_\beta\, d\xi\, ds = 0. \tag{5.6}$$

For the first order approximation, $\delta y_\beta = \delta y_\beta^{(0)}$, so that (5.6) becomes

$$\oint \int_{-1}^{1} [s_{\gamma\alpha}^{(0)} y_{\beta,\gamma}^{(0)}\, {}_0n_\alpha - t_\beta^{(0)}]\, d\xi\, \delta y_\beta^{(0)}\, ds = 0$$

which yields, for the appropriate boundary conditions of the plane stress theory,

$$s_{\gamma\alpha}^{(0)} y_{\beta,\gamma}^{(0)}\, {}_0n_\alpha = \frac{1}{2}\int_{-1}^{1} t_\beta^{(0)}(\xi, s)\, d\xi. \tag{5.7}$$

For the coefficient of h_0^2, equation (5.4) yields

$$\int_{A_e} [(s_{\gamma\alpha}^{(0)}\xi^2 y_{\beta,\gamma}^{(2)} + s_{\gamma\alpha}^{(2)} y_{\beta,\gamma}^{(0)})\,{}_0n_\alpha - t_\beta^{(2)}]\,\delta y_\beta\,d\xi\,ds + \int_{A_e} (s_{\gamma\alpha}^{(0)}\xi Z_{,\gamma}^{(0)}\,{}_0n_\alpha - t_3^{(1)})\,\delta Z\,d\xi\,ds = 0. \qquad (5.8)$$

In (5.8), $\delta y_\beta = \delta y_\beta^{(0)} + h_0^2\xi^2\delta y_\beta^{(2)} = h_0^2\xi^2\delta y_\beta^{(2)}$, since $y^{(0)}$ is known from the first order approximation and hence $\delta y_\beta^{(0)} = 0$. Similarly, $\delta Z = h_0^2\xi^3\delta Z^{(2)}$. Equation (5.8) then yields for the boundary conditions of the second order approximation

$$\begin{aligned} \int_{-1}^{1} \xi^2(\xi^2 s_{\gamma\alpha}^{(0)} y_{\beta,\gamma}^{(2)} + s_{\gamma\alpha}^{(2)} y_{\beta,\gamma}^{(0)})\,{}_0n_\alpha\,d\xi &= \int_{-1}^{1} \xi^2 t_\beta^{(2)}(s,\xi)\,d\xi \\ \int_{-1}^{1} \xi^4 s_{\gamma\alpha}^{(0)} Z_{,\gamma}^{(0)}\,{}_0n_\alpha\,d\xi &= \int_{-1}^{1} \xi^3 t_3^{(1)}(s,\xi)\,d\xi. \end{aligned} \qquad (5.9)$$

The appropriate edge boundary conditions for higher order approximations can be similarly obtained from equation (5.4).

6. INCOMPRESSIBLE MATERIALS

For incompressible materials, expansions (3.7) and (3.9) still hold. The expansions for strain invariants I_1 and I_2 given by (3.10) remain valid, but now $I_3 = 1$ and from (3.10),

$$[C_{11}^{(0)}C_{22}^{(0)} - (C_{12}^{(0)})^2]C_{33}^{(0)} = 1 \qquad (6.1)$$

$$\begin{aligned} &C_{11}^{(0)}C_{22}^{(0)}C_{33}^{(2)} + C_{22}^{(0)}C_{33}^{(0)}C_{11}^{(2)} + C_{33}^{(0)}C_{11}^{(0)}C_{22}^{(2)} \\ &\quad -(C_{13}^{(1)})^2C_{22}^{(0)} - (C_{23}^{(1)})^2C_{11}^{(0)} + 2C_{12}^{(0)}C_{23}^{(1)}C_{13}^{(1)} - 2C_{12}^{(0)}C_{12}^{(2)}C_{33}^{(0)} = 0 \end{aligned} \qquad (6.2)$$

and higher order terms. These are the incompressibility conditions.

The strain energy function W is now a function of I_1 and I_2 only. The expansions for Φ and Ψ given by (3.13) thus take the form

$$\begin{aligned} \Phi_0 &= \Phi_0(I_1^{(0)}, I_2^{(0)}), & \Phi_1 &= \Phi_3 = \Phi_5 = \cdots = 0 \\ \Phi_2 &= BI_1^{(2)} + CI_2^{(2)}, & \Phi_4 &= \cdots \\ \Psi_0 &= \Psi_0(I_1^{(0)}, I_2^{(0)}), & \Psi_1 &= \Psi_3 = \Psi_5 = \cdots = 0 \\ \Psi_2 &= CI_1^{(2)} + EI_2^{(2)}, & \Psi_4 &= \cdots \end{aligned} \qquad (6.3)$$

where B, C and E are the same as before. The invariant function p is now an unknown function of x_i. We assume that p can be expanded into powers of h_0 in the form

$$p(x_1, x_2, \xi; h_0) = p_0(x_1, x_2) + h_0^2\xi^2 p_2(x_1, x_2) + h_0^4\xi^4 p_4(x_1, x_2) + \cdots \qquad (6.4)$$

where $p_0, p_2, p_4 \cdots$ are unknown functions to be determined. The expansions for the constitutive relations (3.14)–(3.16) are valid, except now Φ, Ψ and p are given by (6.3) and (6.4).

The equations of equilibrium for the plane stress theory are still given by (4.4) and (4.5), and the constitutive relations given by (4.6). These equations together with the incompressibility condition (6.1) constitute seven equations for the determination of seven unknowns $s_{\alpha\beta}^{(0)}$, $y_\alpha^{(0)}$, $Z^{(0)}$ and p_0. The edge boundary conditions given by (5.7) remain valid.

Higher order approximations for incompressible materials can be similarly obtained. For example, for the first order interior correction to the plane stress theory, equations of equilibrium (4.7) and (4.8), the constitutive equations (4.9), and the incompressibility condition (6.2) are seven equations for the determination of seven unknowns $s_{\alpha\beta}^{(2)}$, $y_{\alpha}^{(2)}$, $Z^{(2)}$ and p_2. The edge boundary conditions are given by (5.9).

7. SUMMARY

By means of parametric expansions in terms of thickness h_0 of the undeformed plate, the equations of plane stress for finite deformation are obtained as the coefficients of zero power of h_0 in the expansions. The stress components $s_{\alpha 3}^{(0)}$ vanishes identically, and the stress component $s_{33}^{(0)}$ vanishes identically as the natural consequence of surface boundary conditions. The remaining stress components $s_{\alpha\beta}^{(0)}$ are functions of x_α only and are given by

$$s_{\alpha\beta}^{(0)} = \Phi_0\delta_{\alpha\beta} + \Psi_0[\delta_{\alpha\beta}(C_{\rho\rho}^{(0)} + Z^{(0)}Z^{(0)}) - C_{\alpha\beta}^{(0)}] + p_0\bar{C}_{\alpha\beta}^{(0)}. \tag{7.1}$$

The equations of equilibrium have the form

$$(s_{\alpha\beta}^{(0)}y_{\gamma,\beta}^{(0)})_{,\alpha} = 0 \tag{7.2}$$

$$(s_{\alpha\beta}^{(0)}Z_{,\beta}^{(0)})_{,\alpha} = 0. \tag{7.3}$$

The edge boundary conditions are given by

$$s_{\gamma\alpha}^{(0)}y_{\beta,\gamma}^{(0)}\,{}_0n_\alpha = \frac{1}{2}\int_{-1}^{1} t_\beta^{(0)}(\xi, s)\,\mathrm{d}\xi. \tag{7.4}$$

The condition of vanishing of $s_{33}^{(0)}$, that is,

$$0 = \Phi_0 + \Psi_0 C_{\rho\rho}^{(0)} + p_0/(C_{33}^{(0)})^2 \tag{7.5}$$

is an additional condition which must be satisfied so that plane stress theory is truly two-dimensional. When $Z^{(0)}$ varies slowly with respect to x_α, the derivative $Z_{,\alpha}^{(0)}$ may be assumed to vanish identically and equation (7.3) is satisfied automatically. Equation (7.5) then furnishes the relation between $Z^{(0)}$ and $y_\alpha^{(0)}$. In such case, (7.1), (7.2) and (7.5) are basic equations for the plane stress theory.

The coefficients of higher order expansions provide the three-dimensional corrections to the plane stress theory. The first order corrections to the plane stress theory are given by (4.7), (4.8) and (4.9). The appropriate edge boundary conditions are given by (5.9).

REFERENCES

[1] J. E. ADKINS, A. E. GREEN and G. C. NICHOLAS, Two-dimensional theory of elasticity for finite deformations. *Phil. Trans. R. Soc.* **A247**, 279 (1954).
[2] A. E. GREEN and J. E. ADKINS, *Large Elastic Deformations*. Oxford University Press (1960).
[3] E. L. REISS and S. LOCKE, On the theory of plane stress. *Q. appl. Math.* **19**, 195 (1961).
[4] K. O. FRIEDRICHS and R. F. DRESSLER, A boundary-layer theory for elastic plates. *Communs pure appl. Math.* **14**, 1 (1961).
[5] E. REISSNER, On the derivation of boundary conditions for plate theory. *Proc. R. Soc.* **A276**, 178 (1963).
[6] Y. C. FUNG, *Foundations of Solid Mechanics*. Prentice-Hall (1965).

(*Received* 17 *July* 1967; *revised* 22 *April* 1968)

Абстракт—При помощи параметрического разложения по толщине недеформированной пластинки, выводятся уравнения плоского наряженного состояния для конечных деформаций однородных, изотропных, упругих материалов. Уравнения основаны на теории упругости конечных деформаций, без учета обычных геометрических или физических гипотез, кроме предположения ,что деформация яняется симметрической относительно серединной плоскости пластинки. Уравнения плоского напряженного состояния получены приравниванием нулю коэффициентов при нулевых степенях h_0 описанных разложений. Коэффициенты членов высшего порядка дают внутренные поправки к теории плоского напряженного состояния. Выводятся соответствующие граничные условия для этих внутренных уравнений, путем вариационной формулировки трехмерной теории упругости конечных деформаций.